

# Unit 25

## Hypothesis Tests about Proportions

Objectives:

- To perform a hypothesis test comparing two population proportions

Now that we have discussed hypothesis tests to compare means, we want to discuss a hypothesis test to compare proportions. Recall that the  $z$  test statistic that we have used in previous hypothesis tests concerning one population proportion  $\lambda$  was simply the difference between the sample proportion  $\bar{p}$  and the hypothesized proportion  $\lambda_0$  divided by the standard error of the sample proportion  $\sqrt{\lambda_0(1-\lambda_0)/n}$  under the assumption that the hypothesized proportion  $\lambda_0$  is correct. Specifically, this  $z$  test statistic in a hypothesis test about one population proportion is

$$z = \frac{\bar{p} - \lambda_0}{\sqrt{\frac{\lambda_0(1-\lambda_0)}{n}}},$$

The  $z$  statistic used in hypothesis tests to compare two proportions with independent simple random samples of sizes  $n_1$  and  $n_2$  is of the same form. We shall consider only tests where the hypothesized difference between proportions is zero, since this is generally the case in practice (although it need not be). The  $z$  statistic for this hypothesis test will be the difference between the sample proportions  $\bar{p}_1 - \bar{p}_2$  divided by the standard error of the difference between the sample proportions, that is,

$$z = \frac{\bar{p}_1 - \bar{p}_2}{\sqrt{\bar{p}_* \left(1 - \bar{p}_*\right) \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

where  $\bar{p}_*$  is the *combined sample proportion*, that is, the sample proportion we calculate from combining together the two samples. The denominator of this  $z$  statistic estimates the standard error of the difference between the sample proportions under the assumption that the two population proportions are equal, i.e., under the assumption that  $H_0$  is true.

Denoting our two population proportions as  $\lambda_1$  and  $\lambda_2$ , we can write the null hypothesis as  $H_0: \lambda_1 - \lambda_2 = 0$ , which states that the two population proportions are equal. The  $z$  test about two proportions is performed using the same four steps that we have used in each previous hypothesis test. We can verify that our sample sizes are sufficiently large for the  $z$  test to be appropriate by checking that  $n_1 \bar{p}_*$ ,  $n_1(1 - \bar{p}_*)$ ,  $n_2 \bar{p}_*$ ,

and  $n_2(1 - \bar{p}_*)$  are each greater than or equal to 5.

To illustrate, let us use a 0.10 significance level to perform a hypothesis test to see if there is any evidence of a difference between the channel 4 viewing area and the channel 8 viewing area in the proportion of residents who viewed the 11:00 newscast telecast by both channels. A simple random sample of 175 residents in the channel 4 viewing area is selected, and a simple random sample of 225 residents in the channel 8 viewing area is selected. Each resident in each sample is asked whether or not he/she viewed last night's 11:00 newscast.

In the channel 4 sample, 49 residents viewed the newscast, and in the channel 8 sample, 81 residents viewed the newscast.

To verify that our sample sizes are sufficiently large for the  $z$  test to be appropriate, we first find the combined sample proportion  $\bar{p}_*$ . Pooling the samples together, we find that the total sample size is

$175 + 225 = 400$  residents, and the number of these residents who viewed last night's 11:00 newscast is  $49 + 81 = 130$ . Consequently, the combined sample proportion of residents who viewed the newscast is

$\bar{p}_* = 130/400 = 0.325$ . By observing that  $(175)(0.325) = 56.875$ ,  $(175)(0.675) = 118.125$ ,  $(225)(0.325) =$

$73.125$ , and  $(225)(0.675) = 151.875$  are each greater than 5, we verify that the use of the  $z$  test statistic is appropriate. We shall now perform the  $z$  test about two proportions.

The first step is to state the null and alternative hypotheses, and choose a significance level. We shall state the hypotheses in terms of the difference between proportions; the order of subtraction is an arbitrary choice, and we shall choose to subtract the proportion for channel 8 from the proportion for channel 4. Since we are looking for a difference in either direction, this test is two-sided. We complete the first step of the hypothesis test as follows:

$$H_0: \lambda_4 - \lambda_8 = 0 \text{ vs. } H_1: \lambda_4 - \lambda_8 \neq 0 \quad (\alpha = 0.10).$$

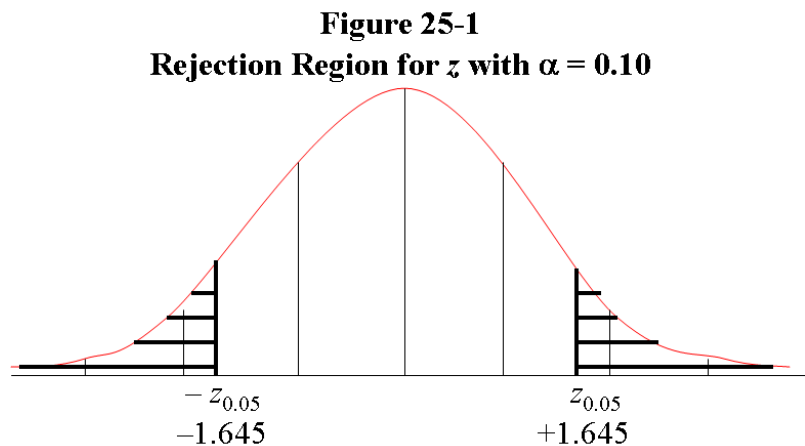
The second step is to collect data and calculate the value of the test statistic. From the data, it is easy to find that the channel 4 sample proportion of residents who viewed the newscast is  $\bar{p}_4 = 49/175 = 0.28$ , and that the channel 8 sample proportion of residents who viewed the newscast is  $\bar{p}_8 = 81/225 = 0.36$ . We have previously found that the combined sample proportion of residents who viewed the newscast is  $\bar{p}_* = 130/400 = 0.325$ . We then calculate the value of our  $z$  test statistic as follows:

$$z = \frac{0.28 - 0.36}{\sqrt{0.325(1 - 0.325)\left(\frac{1}{175} + \frac{1}{225}\right)}} = -1.695.$$

Although we have displayed the calculations to obtain the test statistic here, you can of course avoid having to do the calculations with access to the proper statistical software or programmable calculator.

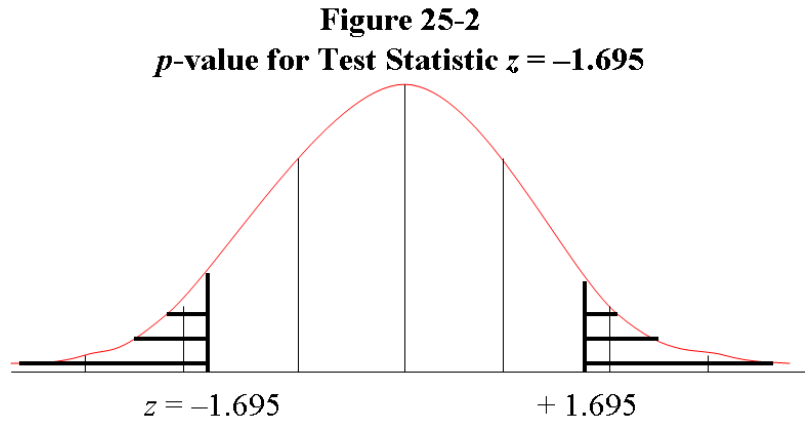
The third step is to define the rejection region, decide whether or not to reject the null hypothesis, and obtain the  $p$ -value of the test. Figure 25-1 displays the rejection region graphically. The shaded area corresponds to values of the  $z$  test statistic where  $\bar{p}_4 - \bar{p}_8$  away from the hypothesized zero difference between proportions and which are unlikely to occur if  $H_0: \lambda_4 - \lambda_8 = 0$  is true. In order that this shaded area be equal to  $\alpha = 0.10$ , the rejection region is defined by the  $z$ -score above which 0.05 of the area lies; below the negative of this  $z$ -score lies 0.05 of the area, making a total area of 0.10. From the last row of Table A.3 ( $df = \infty$ ), or from the bottom of Table A.2, we find that  $z_{0.05} = 1.645$ , and we can then define our rejection region algebraically as

$$z \geq +1.645 \text{ or } z \leq -1.645 \quad (\text{i.e., } |z| \geq 1.645).$$



Since our test statistic, calculated in the second step, was found to be  $z = -1.695$ , which is in the rejection region, our decision is to reject  $H_0: \lambda_4 - \lambda_8 = 0$ ; in other words, our data provides sufficient evidence to suggest that  $\lambda_4 - \lambda_8 \neq 0$  is true.

Figure 25-2 graphically illustrates the  $p$ -value. The  $p$ -value hypothesis test is the probability of obtaining a test statistic value  $z$  which represents a larger distance away from the hypothesized zero difference between proportions than the observed  $z = -1.695$ . In Figure 25-2, the observed test statistic value  $z = -1.695$  and the value  $+1.695$  have both been located on the horizontal axis. The shaded area corresponds to test statistic values which represent a larger distance away from the hypothesized zero difference between means than the observed  $z = -1.695$ ; in other words, the shaded area in Figure 25-2 is the  $p$ -value of the hypothesis test. From the last row of Table A.3 ( $df = \infty$ ), or from the bottom of Table A.2, we find that  $1.695$  is between  $z_{0.05} = 1.645$  and  $z_{0.025} = 1.960$ , which tells us that the area above the  $z$ -score  $1.695$  is between  $0.025$  and  $0.05$ . Therefore, the shaded area in Figure 25-2, which is the  $p$ -value, must be between  $0.05$  and  $0.10$ . We indicate this by writing  $0.05 < p\text{-value} < 0.10$ . The fact that  $0.05 < p\text{-value} < 0.10$  confirms to us that  $H_0$  is rejected with  $\alpha = 0.10$ . However, it also tells us that  $H_0$  would be not be rejected with  $\alpha = 0.05$  nor with  $\alpha = 0.01$ .

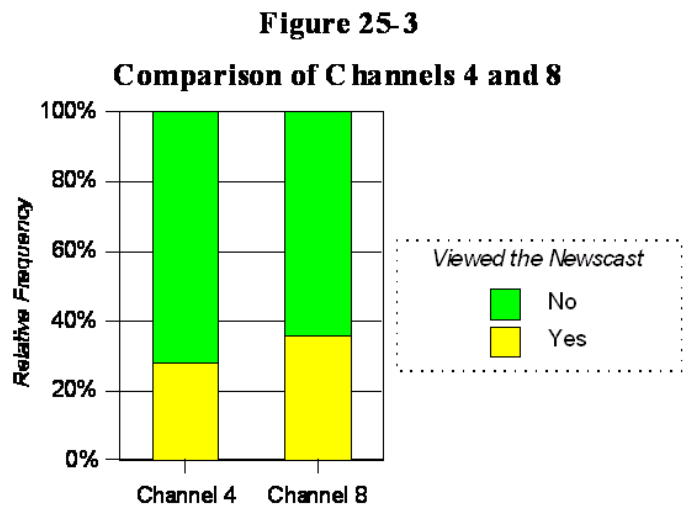


To complete the fourth step of the hypothesis test, we can summarize the results of the hypothesis test as follows:

Since  $z = -1.695$  and  $z_{0.05} = 1.645$ , we have sufficient evidence to reject  $H_0$ . We conclude that there is a difference between the channel 4 and channel 8 viewing areas in the proportion of residents who viewed the 11:00 newscast telecast by both channels ( $0.05 < p\text{-value} < 0.10$ ). The data suggest that the proportion is higher among the channel 8 viewers.

A stacked bar chart can be used to exhibit the data for this hypothesis test. Figure 25-3 is a stacked bar chart with the two different channels labeled on the horizontal axis and the stacks defined by whether or not residents watched the 11:00 newscast.

Up to this point, we have effectively introduced five different hypothesis tests: a one sample  $t$  test, a paired  $t$  test, a two sample  $t$  test, a one sample  $z$  test about a proportion, and a two sample  $z$  test comparing proportions. We shall be expanding this list as we introduce more hypothesis tests later on. However, before introducing more hypothesis tests, we shall consider a type of statistical analysis which is an alternative to hypothesis testing.

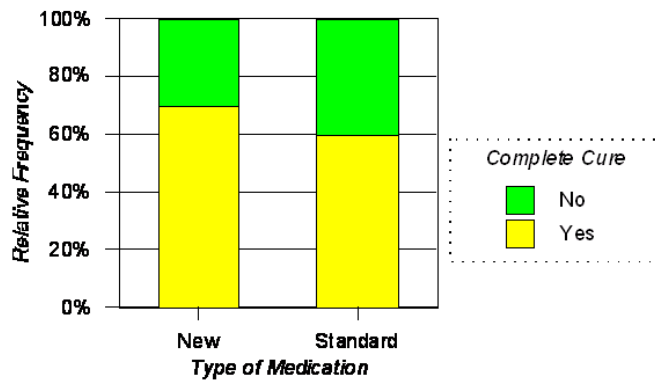




### Answers to Self-Test Problems

- 25-1** (a) Step 1:  $H_0: \lambda_N - \lambda_S = 0$ ,  $H_1: \lambda_N - \lambda_S > 0$ , ( $\alpha = 0.01$ , one-sided)
- Step 2:  $\bar{p}_N = 140/200 = 0.70$ ,  $\bar{p}_S = 180/300 = 0.60$ ,  $\bar{p}_* = 330/500 = 0.64$ , and  $z = +2.282$ .
- Step 3: The rejection is  $z \geq +2.326$ .  $H_0$  is rejected;  $0.01 < p\text{-value} < 0.025$ .
- Step 4: Since  $z = 2.282$  and  $z_{0.005} = 2.326$ , we do not have sufficient evidence to reject  $H_0$ . We conclude that the proportion of complete cures is not higher for the new medication ( $0.01 < p\text{-value} < 0.025$ ).
- (b)  $(200)(0.64) = 128$ ,  $(200)(0.36) = 72$ ,  $(300)(0.64) = 192$ , and  $(300)(0.36) = 108$  are all larger than 5 implying  $n$  is sufficiently large.
- (c) Since  $H_0$  is not rejected, a Type II error is possible, which is concluding that  $\lambda_N - \lambda_S = 0$  when really  $\lambda_N - \lambda_S > 0$ .
- (d)  $H_0$  would have been rejected with  $\alpha = 0.05$  and with  $\alpha = 0.10$ .
- (e) Figure 25-4 is one possible stacked bar chart.

**Figure 25-4**  
**Comparison of Medications**



### Summary

In a test comparing two population proportions, the hypothesized value of the difference between proportions is usually zero. With two sufficiently large independent simple random samples, this hypothesis test can be performed by using the test statistic

$$z = \frac{\bar{p}_1 - \bar{p}_2}{\sqrt{\bar{p}_* \left(1 - \bar{p}_*\right) \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

where  $\bar{p}_*$  is the *combined sample proportion*, that is, the sample proportion we calculate from combining together the two samples of sizes  $n_1$  and  $n_2$ . An easy way to verify that the sample sizes are sufficiently large for the  $z$  test to be appropriate is by checking that  $n_1 \bar{p}_*$ ,  $n_1(1 - \bar{p}_*)$ ,  $n_2 \bar{p}_*$ , and  $n_2(1 - \bar{p}_*)$  are each greater than or equal to 5.